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On Definition of Skew Frames

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Abstract Skew frames represent a common generalization of frames and orthomodular lattices. They could serve as Lindenbaum algebras of quantum intuitionistic logic as well as invariants of noncommutative C*-algebras. It is shown that lattices of open projections with skew (partial) operations are complete invariants of C*-algebras and that these operations are preserved by morphims of C*-algebras.

Keywords Intuitionistic quantum logic · Noncommutative topology · Orthomodular lattice · Skew lattice

1 Introduction

From the time of Birkhoff and von Neumann pioneer work [5], orthomodular lattices represent quantum logic, and thus can be considered as "quantized" Boolean algebras. In contrast to Boolean algebras, orthomodular lattices are not longer distributive and the lattice operations are not residuated. On the other hand, the binary joins and meets can be replaced by skew joins and skew meets [4] defined, respectively, by assignments

$$x \mathbin{\dot\wedge} y = x \wedge (x' \vee y),$$
$$x \mathbin{\dot\vee} y = x \vee (x' \wedge y).$$

(The unary version $\phi_x(y) = x \mathbin{\dot\wedge} y$ for fixed x is known as *Sasaki projection*.) Assume, for simplicity, that we are already dealing with a complete orthomodular lattice M. It holds that

$$x \mathbin{\dot\wedge} \bigvee y_i = \bigvee (x \mathbin{\dot\wedge} y_i)$$

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(as well as its De Morgan dual) but, in general $(\bigvee y_i)\,\dot{\wedge}\,x \neq \bigvee(y_i\,\dot{\wedge}\,x)$. Thus, the skew meet is residuated on the left, i.e. $\phi_x = x\,\dot{\wedge}\,(-)$ has a right adjoint $\phi^x = x \to (-)$, called *Sasaki hook*, which is given by $x \to y = x' \vee (x \wedge y) = x'\,\dot{\vee}\,y$ [9]. It means that

$$x\,\dot{\wedge}\,y \leq z \;\Leftrightarrow y \leq x \to z$$

holds for any elements $x, y, z \in M$. The negative property of the skew operations is that they are no more commutative nor associative.

Kröger proved in [10] that orthomodular lattices can be characterized alternatively in terms of skew operations as *Boolean skew lattices*. The interest in skew operations is motivated at least by the following reasons:

- The adjunction between Sasaki operations admits the modus ponens law $x\,\dot{\wedge}\,(x \to y) \leq y$ which is necessary for logic.
- The skew meet corresponds to multiplication of projections on a Hilbert space. In contrast to other approaches, e.g. effect algebras, skew operations are defined globally.

Another way of generalizing Boolean algebras is the lack of De Morgan duality. Assuming again completeness, the resulting structures are complete Heyting algebras or frames (depending on considered operations). These structures represent both intuitionistic logic and topology. Let us recall that every frame (even not spatial) is a subframe of a complete Boolean algebra [8].

The quantized version of intuitionistic logic was studied by many authors, e.g. [3, 6, 11], and it is also very closed to the problem of spectra of noncommutative C*-algebras [1, 2, 7, 14]. Our aim is to combine properties of Kröger's skew Boolean algebras with those of frames. The resulting structure is called a *skew frame*. Following the analogy with frames one can expect that skew frames should be substructures of complete orthomodular lattices closed under joins and skew operations. In order to obtain some intuition of the concept we will follow the relations with C*-algebras.

Lattices/algebras	Classical	Quantum
De Morgan	(complete) Boolean algebras / commutative W*-algebras	(complete) orthomodular lattices / W*-algebras
intuitionistic	frames / commutative C*-algebras	skew frames / C*-algebras

Recall [15] that the (complete orthomodular) lattice of projections $P(B)$ of a W*-algebra B is isomorphic to the lattice $r(B)$ of right ideals closed in normal topology. Given a C*-algebra A, one can consider its reduced atomic representation $\pi : A \to B$ to its enveloping W*-algebra B. In a more detail, $B = \prod \mathcal{B}(H_i)$ is a product of operator algebras on Hilbert spaces H_i and π consists of pairwise non-equivalent irreducible representations $\pi_i : A \to \mathcal{B}(H_i)$. The representation yields a top-preserving sup-lattice embedding $\mu : R(A) \to r(B)$ from the lattice $R(A)$ of norm-closed right ideals of A to the lattice $r(B)$ of w*-closed right ideals of B [2, 7]. The projections of $P(B)$ associated to the image elements of $R(A)$ are called *open projections*. Akemann, Giles, and Kummer proved that the structure of open projections $P(A)$, considered as a substructure of $P(B)$, determines the original C*-algebra A. It remains to answer the question whether $R(A)$ alone determines the mapping μ. Rosický noted [14] that the algebra need not be determined even if $R(A)$ is considered as a *quantum frame*, i.e. a sup-lattice with multiplication $r \circ s = r^* s$. We will show that the skew frame structure should carry the complete information about the C*-algebra A and would be a much simpler complete invariant of C*-algebras than Max A introduced by Mulvey and Pelletier [12].

2 The Structure of Open Projections

Let $\pi : A \to B, \mu : R(A) \to r(B)$ be representations mentioned above. Thus $B = \prod \mathcal{B}(H_i)$ is a product of operator algebras on Hilbert spaces H_i and π consists of pairwise non-equivalent irreducible representations $\pi_i : A \to \mathcal{B}(H_i)$. (Since π can be considered as an inclusion, we will not distinguish elements $a \in A$ and $\pi(a) \in B$.) Following [7], a projection $p \in P(B)$ is *open* if there is a positive element $a \in A^+$ with *support* p, i.e. p is the lowest projection such that $pa = a$. When

$$a = \int_{\lambda \in \mathbb{R}^+} \lambda \, dE(\lambda)$$

is a spectral resolution of a then

$$p = \int_{\lambda \in \mathbb{R}^+} dE(\lambda).$$

By functional calculus we have

$$a^{1/n} = \int_{\lambda \in \mathbb{R}^+} \lambda^{1/n} dE(\lambda)$$

and hence p can be calculated (in B) as a pointwise (strong operator) limit $p = \text{s-lim}\, a^{1/n}$ of elements of A. Hence we will denote p by a^0. When $\|a\| \leq 1$ then this sequence is increasing and we obtain an approximate projection $(a^{1/n})$ for p in A. This idea enables to consider projections as formal limits of elements from A not mentioning the reduced atomic representation.

It is well known that the skew operations on $P(B)$ can be defined as follows

$$p \dot{\wedge} q = (pqp)^0,$$
$$p \dot{\vee} q = 1 - (1-p) \dot{\wedge} (1-q).$$

Akemann [2] proved that a meet of two open projections need not be open. However, the author assume that the following property holds:

Conjecture 1 *If p, q are open then $p \dot{\wedge} q$, $p \dot{\vee} q$ are open.*

Let $f : A_1 \to A_2$ be a morphism of C*-algebras. Since f preserves positive elements and their roots, we have $\text{s-lim}\, f(a^{1/n}) = \text{s-lim}\, f(a)^{1/n}$, and hence we can correctly extend f to open projections by setting $f(a^0) = f(a)^0$. Since the multiplication is continuous in the strong operator topology, we have also $f(a^0 b^0) = f(a^0) f(b^0)$. Hence,

$$f(p) \dot{\wedge} f(q) = [f(p) f(q) f(p)]^0 = f(pqp)^0 = f[(pqp)^0] = f(p \dot{\wedge} q)$$

for open projections p, q of A_1. Similarly, when A is unital, then the *closed* (i.e. complementary to open) projections $1 - p = 1 - a^0$, $1 - q = 1 - b^0$ are also strong operator limits of sequences $(1 - a^{1/n}), (1 - b^{1/n})$, respectively, thus f preserves also their skew meet. Consequently, the open projections p, q satisfy

$$f(p \dot{\vee} q) = f(p) \dot{\vee} f(q).$$

If A_1, A_2 are not unital, then we obtain the result by standard adjoining the units.

Assuming Conjecture 1, we obtain that the sup-lattice of open projections $P(A)$ can be equipped by skew operations which are preserved by sup-lattice morphisms induced by C*-algebra morphisms. In other words, the assignment $A \mapsto P(A)$ would yield a functor from C*-algebras to skew frames. Notice that when Conjecture 1 is not true, then the skew operations can be still defined partially (at least when the projections commute).

As mentioned in Introduction, the sup-lattice $P(A)$ is not sufficient to reconstruct the embedding $P(A) \to P(B)$ but we can show now that the skew structure (even considered partial) is reach enough. Notice that the maximal elements of $P(A)$ bijectively correspond to pure states of A and the equivalence of pure states can be easily recovered as a transitive cover of the non-orthogonality relation. That is, we say that pure states are orthogonal if their opposite one-dimensional (closed) projections in B are orthogonal. But this precisely means that the open projections p, q associated to the pure states satisfy $p \dot{\vee} q = 1$. From the equivalence and orthogonality of pure states we can restore the Hilbert spaces H_i and consequently the orthomodular lattice $P(B)$. Finally, for an open projection p and maximal open projection q associated to pure state ϕ we have $p \to q = (p \dot{\wedge} q')'$ in $P(B)$ but $p \to q$ is also maximal or the top (namely support of the kernel of $\phi(p(-)p)$) and hence open, i.e. $p \to_{P(A)} q = p \to_{P(B)} q$. Thus the action of p on $P(B)$ is completely determined by its action on pure states. We have recovered the embedding $P(A) \to P(B)$ and, following Akemann, Giles, and Kummer, we can recover A.

3 Properties of Skew Frames and Open Problems

Following the fundamental example of open projections, one can expect that every skew frame can be embedded into a complete orthomodular lattice. Notice also that a central cover of any open projection is open [7], hence the skew frame should be closed under the central cover, denoted by $|-|$. We are ready to formulate the definition.

Definition 1 A sup-lattice S is called a *skew frame* if it is sub-sup-lattice of a complete orthomodular lattice M, containing the top element 1_M and closed under skew joins, skew meets, and the central cover, i.e. if $a, b, a_i \in S$ then $1_M, a \dot{\wedge} b, a \dot{\vee} b, |a|, \bigvee a_i \in S$.

It follows that skew frames satisfy any identity valid in complete orthomodular lattices which is expressed by operations $\bigvee, \dot{\wedge}, \dot{\vee}, |-|$ (but not those with orthocomplement $'$). The problem is to select a worth axiomatic system. From properties

$$|x \dot{\wedge} y| = |y \dot{\wedge} x|,$$

$$|x \dot{\wedge} z| = |x \wedge z| = |x| \wedge z$$

for $x, y \in M, z \in Z(M)$ (the center of M) [9] it follows immediately that a skew frames admit the quantum frame operation $x \circ y = |x \dot{\wedge} y|$. The second above property can be also interpreted as a Frobenius condition, hence the embedding $Z(M) \cap S \to S$ is *open* in the usual categorical sense (with the central cover map as its left adjoint).

The further problem appears when we forget about the parental orthomodular lattice and we want to re-represent the skew frame. Following the classical situation, i.e. the embedding of a frame into a complete Boolean algebra, one should introduce a concept of *nuclei* for skew frames. A very similar subject on orthomodular lattices was studied by Román [13]. However, we can expect also that there is a class of *spatial* skew frames, for which the representation should be easier.

A crucial problem for the theory is a proof of Conjecture 1.

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